Transition between two oscillation modes

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A model for the symmetric coupling of two self-oscillators is presented. The nonlinearities cause the system to vibrate in two modes of different symmetries. The transition between these two regimes of oscillation can occur by two different scenarios. This might model the release of vortices behind circular cylinders with a possible transition from a symmetric to an antisymmetric Bénard–von Karman vortex street.
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Nowadays the understanding of self-oscillators is fairly complete thanks to the theory of bifurcation and of normal forms. A familiar model for this is the van der Pol system [1], which displays a wide range of behavior, from weakly nonlinear to strongly nonlinear relaxation oscillations, making it a good model for many practical situations. However, there are physical situations characterized by spontaneous self-oscillations with certain basic features that are absent from the ''generic'' van der Pol system. Take for instance the Bénard–von Karman vortex street in the wake of a cylinder. Its phenomenology is approximately as follows [2–4]: the velocity field remains two-dimensional for Reynolds number (Re) less than 160 (creeping flow for Re<4; recirculation zone with two steady symmetric eddies attached behind the cylinder for 4<Re<45; instability at Re≃45 at which these eddies are released alternatively to form a double row of opposite sign vortices, the Bénard–von Karman vortex street) and for Re>160 three-dimensional and irregular fluctuations are superimposed on the dominant periodic vortex shedding.

It is tempting to say that the periodic vortex shedding provides a classical example of Poincaré-Andronov bifurcation to a limit cycle. However, one fundamental ingredient would be missing if one insisted in describing these oscillations by the van der Pol equation: no equivalent of the symmetry of the system would be present in the mathematical description. That is, this mathematical picture would make no difference between a symmetric and an antisymmetric release of vortices, both would be fairly described by the same van der Pol equation, although they are clearly physically different.

We propose here to implement the major symmetries of the Bénard–von Karman oscillations by assuming that they result from the symmetric coupling of two identical oscillators, each one responsible for the periodic release of vortices on one side of the cylinder. The interest of this approach is that it shows two possible stable oscillating states: one symmetric, one antisymmetric, depending on the value of some coupling parameter. By varying continuously the coupling, it is possible to monitor the transition between these two regimes, something that is beyond an approach using a single van der Pol equation.

A dynamical model for representing these properties is presented and some information about the transitions between the two oscillation modes is obtained. We make no attempt to relate our model to the fluid mechanical equations. The symmetry properties of the system are used as a basic ingredient, as well as the fact that it operates in a stable way in an oscillating mode.

Let us assume that there is an oscillator (the ''vortex'' emitter) on each side (side_1,side_2) of the cylinder [5] and let (x_1,x_2) be their amplitudes of oscillation. If $(x_1(t),x_2(t))$ is a possible dynamics, $(x_2(t),x_1(t))$ is also realizable by symmetry. If a vortex is emitted from side_i whenever $x_i(t)$ reaches a maximum, then the symmetrical and antisymmetrical vortex streets (Fig. 1) appear as two oscillation modes, one with the two oscillators in phase and the other one with the two oscillators out of phase. The symmetric mode Θ_1 verifies $x_1(t)=x_2(t)$ and the antisymmetric one Θ_2 verifies $x_1(t)=x_2(t+T/2)$ with T the period.

The simplest model representing these properties is a system of two coupled harmonic oscillators with a small coupling β:

$$\begin{aligned}\ddot{x}_1+x_1+\beta x_2&=0,\\ \ddot{x}_2+x_2+\beta x_1&=0.\end{aligned}\tag{1}$$

The normal modes verify $\ddot{\Theta}_i+[1-(-1)^i\beta]\Theta_i=0$ ($i=1,2$), with $\Theta_1=x_1+x_2$ and $\Theta_2=x_1-x_2$ each one with frequencies $\omega_1=1+\beta$ and $\omega_2=1-\beta$. But this model is Hamiltonian and it is not useful to describe self-oscillations (i.e., oscillations resulting from balance between energy in-

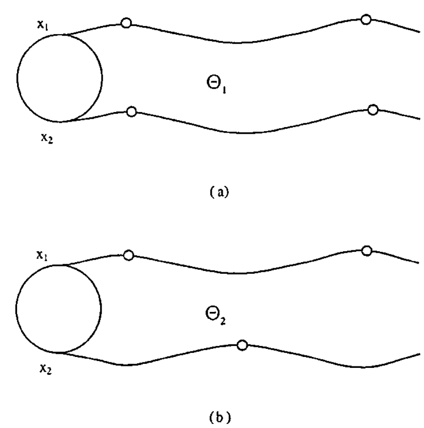

FIG. 1. A graphical representation of the two oscillation modes: (a) the symmetric vortex street Θ_1 and (b) the antisymmetric one Θ_2.

put and dissipation). It does not present any Poincaré-Andronov bifurcation, although it was shown experimentally that the vortex shedding behind a cylinder results from that type of bifurcation [2]. To remedy this, we can introduce, as in the van der Pol system, a nontrivial damping term $\dot{x}f(x)$ in Eqs. (1). The van der Pol oscillator of equation $\ddot{x}-\epsilon(1-x^2)\dot{x}+x=0$ is probably the simplest example of a system with one stable limit cycle: a fixed point at the origin and an unstable closed orbit for $\epsilon<0$ and an attractive cycle for $\epsilon>0$. A natural extension of the van der Pol system to the representation of two symmetric coupled oscillators is

$$\begin{aligned}\ddot{x}_1-\epsilon[1-(x_1^2+x_2^2)]\dot{x}_1+x_1+\beta x_2=0,\\ \ddot{x}_2-\epsilon[1-(x_1^2+x_2^2)]\dot{x}_2+x_2+\beta x_1=0.\end{aligned} \tag{2}$$

The small displacements near $x_1=x_2=\dot{x}_1=\dot{x}_2=0$ are damped to zero when $\epsilon<0$ and give sustained oscillations when $\epsilon>0$. The birth of a stable limit cycle is then governed by the parameter ϵ and our interest rests in this regime ($\epsilon>0$). Other properties of Eqs. (2) are as follows.

(i) If $\beta=0$, Eqs. (2) present $O(2)$ symmetry. If the complex variable $z=x_1+ix_2$ is defined, system (2) become $\ddot{z}-\epsilon(1-|z|^2)\dot{z}+z=0$, which has the symmetries $z\rightarrow e^{i\phi}z$ and $z\rightarrow\bar{z}$. A stable solution is $z=e^{i\phi}r(t)$ with ϕ constant and $r(t)$ the solution of $\ddot{r}-\epsilon(1-r^2)\dot{r}+r=0$. Its representation point is a straight line through the origin in the (x_1,x_2) plane at constant angle ϕ, $r(t)$ oscillating along this line. The periodic solution $z=e^{\pm it}$ is unstable.

(ii) If $\beta\neq0$ the phase symmetry is destroyed, although a $Z(2)$ symmetry $z\rightarrow\pm i\bar{z}$ remains. Equations (2) become

$$\ddot{z}-\epsilon(1-|z|^2)\dot{z}+z+i\beta\bar{z}=0. \tag{3}$$

Now there are two oscillating solutions: the symmetric mode $\Theta_1\equiv(x_1=x_2)$ given by $z_1=e^{i\pi/4}r_1(t)$ and the antisymmetric mode $\Theta_2\equiv(x_1=-x_2)$ given by $z_2=e^{-i\pi/4}r_2(t)$, where $r_1(t)$ and $r_2(t)$ verify $\ddot{r}_i-\epsilon(1-r_i^2)\dot{r}_i+[1-(-1)^i\beta]r_i=0$ $(i=1,2)$.

This system presents stable oscillations when $|\beta|<1$ and diverges to infinity when $|\beta|>1$ (except for initial conditions $r=\dot{r}=0$). One finds a parameter value $\beta_\epsilon>0$ such that if $0<\beta<\beta_\epsilon$, then Θ_1 is stable and Θ_2 unstable, and if $-\beta_\epsilon<\beta<0$, Θ_1 is unstable and Θ_2 stable. Also, if $\beta_\epsilon<|\beta|<1$, the two modes Θ_1 and Θ_2 are linearly stable. Let us remark that this simple model brings all the information we are looking for. The range of parameters $\epsilon>0$ would modelize the situations of stable limit cycle oscillations observed experimentally for $\mathrm{Re}>\mathrm{Re}_c$, where Re_c is the Reynolds number at the onset of vortex shedding. The parameter β would represent for instance the aspect ratio in the experiments of Le Gal and collaborators [6].

In model equation (2) the transition from Θ_1 to Θ_2 stable oscillation occurs at $\beta=0$. In this case the coupling is lost and the system becomes degenerate at transition (the phase difference between the two oscillators is arbitrary) presenting an infinity of stable oscillating states. In order to remove this degeneracy, we need to have more than one coupling parameter. This means that the dimension of parameter space for a transition between two modes of oscillation should be greater than 1: the unfolding of this transition should be controlled by two parameters at least. Thus, a more general and "robust" scheme of transition from mode Θ_1 to mode Θ_2, and vice versa, is achieved by introducing another phase symmetry breaking term (proportional to γ) in the dissipative force of Eqs. (2):

$$\begin{aligned}\ddot{x}_1-\epsilon[1-x_1^2-(1+\gamma)x_2^2]\dot{x}_1+x_1+\beta x_2=0,\\ \ddot{x}_2-\epsilon[1-x_2^2-(1+\gamma)x_1^2]\dot{x}_2+x_2+\beta x_1=0,\end{aligned} \tag{4}$$

where γ and β are the coupling constants. Symmetries $(x_1,x_2)\leftrightarrow(x_2,x_1)$ and $(x_1,x_2,\beta)\leftrightarrow(x_2,-x_1,-\beta)$ are preserved.

We have numerically studied the solutions of this system in two different regimes and found the following results for γ and β near zero.

(a) When $|\beta|\ll|\gamma|$ there are four oscillatory states: the pure symmetric mode $\Theta_1\equiv(x_1=x_2)$, the pure antisymmetric mode $\Theta_2\equiv(x_1=-x_2)$, and two new mixed modes $\Theta_{12}\equiv(x_1,x_2)$ and $\Theta_{21}\equiv(x_2,x_1)$ intermediate between Θ_1 and Θ_2. If $\gamma>0$, the mixed modes are stable and the pure modes unstable. If $\gamma<0$, the mixed modes are unstable and the pure modes stable.

(b) When $|\beta|\gg|\gamma|$, Eqs. (4) tend to Eqs. (2) (the perturbation introduced by γ can be neglected in front of β), the mixed modes Θ_{12} and Θ_{21} collide and disappear, and the pure mode Θ_1 and Θ_2 remain.

Let us explain in more detail the two different scenarios (Fig. 2) that can be found for the transition between the pure modes Θ_1 and Θ_2 when γ is fixed and β is varied (ϵ is kept constant and of order 1, but the results are not sensitive to its specific value).

Scenario I, $\gamma<0$ [Fig. 2(a)]. (I_1) $\beta<-c(\epsilon)|\gamma|$ [$c(\epsilon)$ positive constant, depending on ϵ and of order 1 for ϵ of order 1]: Θ_1 is unstable and Θ_2 stable. No mixed modes. (I_2) $-c(\epsilon)|\gamma|<\beta<c(\epsilon)|\gamma|$: the two unstable mixed modes Θ_{12} and Θ_{21} grow from Θ_1 for $\beta=-c(\epsilon)|\gamma|$. In this regime the two pure modes are stable. Depending on initial conditions the system oscillates in the symmetric or in the antisymmetric mode. When $\beta\rightarrow c(\epsilon)|\gamma|$ the two mixed modes approach Θ_2 and collide with it for $\beta=c(\epsilon)|\gamma|$ making Θ_2 linearly unstable. It transfers the stability from Θ_2 to Θ_1. (I_3) $\beta>c(\epsilon)|\gamma|$: Θ_1 is stable and Θ_2 unstable. No mixed modes. Summarizing: there is a range of parameters I_2 where Θ_1 and Θ_2 are both stable, and each mode (Θ_1 or Θ_2) loses its stability by a supercritical bifurcation on the edges of I_2.

Scenario II, $\gamma>0$ [Fig. 2(b)]. (II_1) $\beta<-c(\epsilon)|\gamma|$: Θ_1 is unstable and Θ_2 stable. No mixed modes. (II_2) $-c(\epsilon)|\gamma|<\beta<c(\epsilon)|\gamma|$: the two mixed modes Θ_{12} and Θ_{21} bifurcate from Θ_2 for $\beta=-c(\epsilon)|\gamma|$. These are stable (which makes the difference with scenario I). In this II_2 regime the two pure modes are unstable and the system will decay in one of the two mixed modes according to the initial conditions. When $\beta\rightarrow c(\epsilon)|\gamma|$ the mixed modes approach Θ_1 and collide with it for $\beta=c(\epsilon)|\gamma|$. It transfers the stability from the mixed modes to Θ_1. (II_3) $\beta>c(\epsilon)|\gamma|$. Θ_1 is stable and Θ_2 unstable. No mixed modes. Summarizing: there is a range of parameters II_2 where the two mixed modes are stable, and collide with Θ_1 or Θ_2 on the edge of II_2 to exchange stability.

A derivation of the dynamics of Eqs. (4) can be obtained in the formalism of a slow phase dynamics [7]. [A different

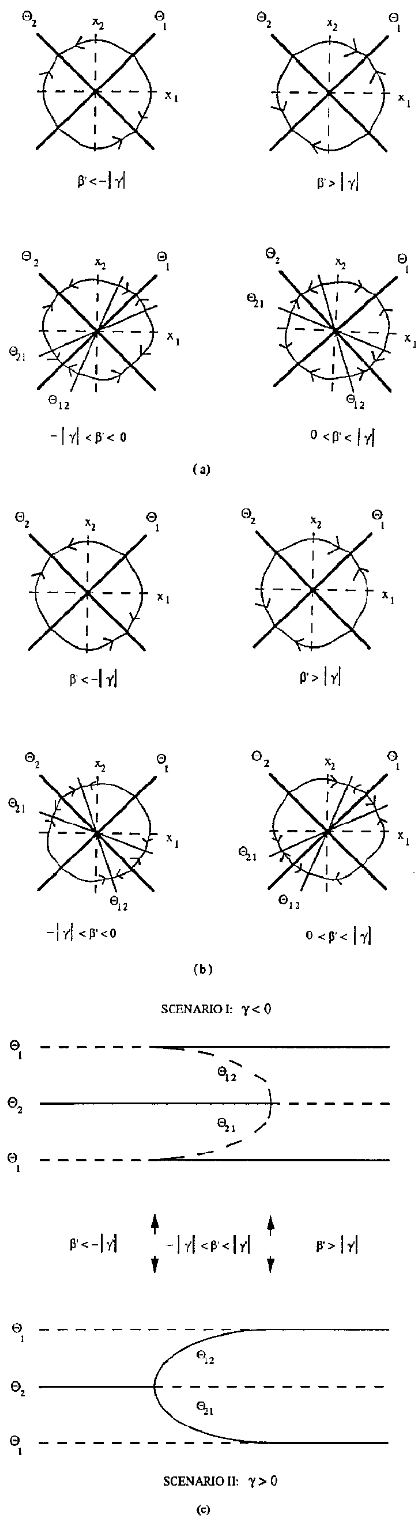

FIG. 2. Nonlinear transition between the two oscillation modes (Θ_1,Θ_2): (a) in scenario I $(\gamma<0)$ the two intermediary mixed modes, Θ_{12} and Θ_{21}, are unstable and (b) in scenario II $(\gamma>0)$ these mixed modes are stable. (c) Another representation of scenarios I and II (inspired from figure 3 in Ref. [8]).

calculation can be found in Ref. [8], where two nonlinear oscillators with diffusive coupling, not the one we consider, are studied in the vicinity of a Hopf (Poincaré-Andronov) bifurcation.] When $\beta=\gamma=0$, the set of Eqs. (4) presents phase symmetry (ϕ) and temporal translation symmetry (ψ), and when β or γ are different from zero the phase symmetry is broken. For small β or γ (of the same order of magnitude) the general solution of Eqs. (4) can be written

$$x_1 = r_0(t+\psi)\cos\phi + \delta x_1,$$

$$x_2 = r_0(t+\psi)\sin\phi + \delta x_2,$$

where $r_0(t)$ is the periodic nonzero solution of the van der Pol equation: $\ddot{r}_0 - \epsilon(1-r_0^2)\dot{r}_0 + r_0 = 0$, ψ and ϕ follow a slow dynamics $(\dot\psi/\psi, \dot\phi/\phi \ll \dot{r}_0/r_0)$, and δx_1, δx_2, $\dot\psi$ and $\dot\phi$ are small and of order (γ,β) Linearizing Eqs. (4) to order (γ,β) we obtain a set of coupled equations to be solved for δx_1 and δx_2. These equations are written in matrix notation to make their structure more transparent:

$$\mathcal{L}\begin{pmatrix}\delta x_1\\ \delta x_2\end{pmatrix} = \begin{pmatrix} f(r_0)\sin\phi & g(r_0)\cos\phi \\ -f(r_0)\cos\phi & g(r_0)\sin\phi\end{pmatrix}\begin{pmatrix}\dot\phi\\ \dot\psi\end{pmatrix} - r_0\left[\beta + \frac{\gamma}{2}\,h(r_0,\phi)\right]\begin{pmatrix}\sin\phi\\ \cos\phi\end{pmatrix}, \tag{5}$$

where

$$\mathcal{L} = \begin{pmatrix}\mathcal{F}_t & 0\\ 0 & \mathcal{F}_t\end{pmatrix} + h(r_0,\phi)\begin{pmatrix}\cot\phi & 1\\ 1 & \tan\phi\end{pmatrix},$$

$$\mathcal{F}_t = \partial_{tt} - \epsilon(1-r_0^2)\partial_t + 1,$$

$$f(r_0) = 2\dot{r}_0 - \epsilon(1-r_0^2)r_0,$$

$$g(r_0) = -[2\ddot{r}_0 - \epsilon(1-r_0^2)\dot{r}_0],$$

$$h(r_0,\phi) = \epsilon r_0\dot{r}_0\sin(2\phi).$$

The relevant solution of Eq. (5) is made of periodic functions of time, with the same period T as $r_0(t)$. This excludes functions with a secular growth and leads to a solvability condition that will ultimately become an equation of evolution for $\phi(t)$. To write this solvability condition, one needs to define first an inner product of functions of time with period T as $\langle\vec\theta|\vec\sigma\rangle \equiv \int_0^T(\theta_1\sigma_1+\theta_2\sigma_2)dt$ [$\vec\theta=(\theta_1,\theta_2)$ is written as a two-component vector]. One notices now that the linear operator $\mathcal{L}$ has a nonempty kernel:

$$\mathcal{L}\vec\omega = \vec{0} \Rightarrow \vec\omega_a = r_0\begin{pmatrix}\sin\phi\\ -\cos\phi\end{pmatrix}, \quad \vec\omega_b = \dot{r}_0\begin{pmatrix}\cos\phi\\ \sin\phi\end{pmatrix}.$$

Because of this nonempty kernel, Eq. (5) has no solution in general that is periodic with period T. To have such a solution, the right-hand side $\vec\varphi$ of this equation must be orthogonal to the kernel of the adjoint operator $\mathcal{L}^+$, made of two functions, $\vec\chi_i$, $(i=a,b)$, of t that are solution of the formal equation $\mathcal{L}^+\vec\chi = 0$. The solvability condition is then that the two inner products $\langle\vec\chi_i|\vec\varphi\rangle$ $(i=a,b)$ are zero. The operator $\mathcal{L}^+$ can be written explicitly as

$$\mathcal{L}^+ = \begin{pmatrix}\mathcal{F}_t^+ & 0\\ 0 & \mathcal{F}_t^+\end{pmatrix} + h(r_0,\phi)\begin{pmatrix}\cot\phi & 1\\ 1 & \tan\phi\end{pmatrix},$$

where $\mathcal{F}_t^+ = \partial_{tt} + \epsilon\partial_t(1-r_0^2) + 1$. Since the two left vectors $\vec\chi_{a,b}$, once multiplied with the inner product, $\langle|\rangle$, with the left side of Eq. (5) give zero, the same product with the right side of Eq. (5) should give zero as well. This gives two coupled equations for $\dot{\phi}$ and $\dot{\psi}$:

$$\begin{pmatrix} h_{a1} & h_{a2} \\ h_{b1} & h_{b2} \end{pmatrix}\begin{pmatrix} \dot{\phi} \\ \dot{\psi} \end{pmatrix} = \beta\begin{pmatrix} m_a \\ m_b \end{pmatrix} + \gamma\begin{pmatrix} n_a \\ n_b \end{pmatrix}, \tag{6}$$

where h_{i1}, h_{i2}, m_i, n_i $(i=a,b)$ are functions of ϵ and ϕ after the time integration coming from the scalar product:

$$h_{i1} = \int_0^T f(r_0)[\sin\phi\,\chi_{i1} - \cos\phi\,\chi_{i2}]dt,$$

$$h_{i2} = \int_0^T g(r_0)[\cos\phi\,\chi_{i1} + \sin\phi\,\chi_{i2}]dt,$$

$$m_i = \int_0^T r_0[\sin\phi\,\chi_{i1} + \cos\phi\,\chi_{i2}]dt,$$

$$n_i = \frac{1}{2}\int_0^T r_0 h(r_0,\phi)[\sin\phi\,\chi_{i1} + \cos\phi\,\chi_{i2}]dt.$$

The ϕ dependence of the vectors in the kernel of $\mathcal{L}^+$ can be factored out by noticing that these vectors can have the following ϕ dependence:

$$\vec{\chi}_a = h_a(t)\begin{pmatrix} -\sin\phi \\ \cos\phi \end{pmatrix}, \quad \vec{\chi}_b = h_b(t)\begin{pmatrix} \cos\phi \\ \sin\phi \end{pmatrix}.$$

From this the two functions $h_{a,b}(t)$ are the nontrivial (=nonzero) solutions of period T of the two linear homogeneous equations:

$$\mathcal{F}_t^+[h_a(t)] = 0 \Rightarrow [\partial_{tt} + \epsilon\partial_t(1-r_0^2) + 1]h_a(t) = 0,$$

$$[\mathcal{F}_t + 2\epsilon(1-r_0^2)\partial_t][h_b(t)]$$
$$= 0 \Rightarrow [\partial_{tt} + \epsilon(1-r_0^2)\partial_t + 1]h_b(t) = 0.$$

Then the equation for $\dot{\phi}$ is simplified to

$$k_a(\epsilon)\dot{\phi} = l_a(\epsilon)\cos(2\phi)\beta + s_a(\epsilon)\sin(4\phi)\gamma, \tag{7}$$

where k_a, l_a, and s_a are functions of ϵ only that are proportional to various scalar product of functions on Eq. (6) with $h_a(t)$. Thus $k_a(\epsilon) = -\int_0^T f(r_0)h_a(t)dt$; $l_a(\epsilon) = \int_0^T r_0 h_{a,b}(t)dt$; and $s_a(\epsilon) = \epsilon/4 \int_0^T r_0^2 \dot{r}_0 h_{a,b}(t)dt$.

Equation (7) presents, as β and γ vary, the same bifurcations as the one found numerically for the original set of equations (4). To show this, let us define $\beta' = \beta[l_a(\epsilon)/k_a(\epsilon)]$ and $\gamma' = 2\gamma[s_a(\epsilon)/k_a(\epsilon)]$, which will be considered now as the bifurcation parameters [the quantities $l_a(\epsilon)/k_a(\epsilon)$ and $s_a(\epsilon)/k_a(\epsilon)$ are constants of order 1 at a fixed finite value of ϵ, and so can be eliminated by scaling]. The fixed points of Eq. (7) are roots (in ϕ) of

$$\beta'\cos(2\phi) + (\gamma'/2)\sin(4\phi) = 0 \tag{8}$$

or of $\cos(2\phi)=0$ or $\beta' + \gamma'\sin(2\phi) = 0$.

If $|\beta'| > |\gamma'|$, this corresponds to scenario $\mathrm{I}_{1,3}$ and $\mathrm{II}_{1,3}$. The only steady states are at the zeros of $\cos(2\phi)$, which are at $\phi = \pi/4$ and $\phi = -\pi/4$, with one stable and the other unstable, depending of the sign of β' (and consequently of β) in agreement with what was found numerically. If $|\beta'| < |\gamma'|$, they are two more fixed points, which are $\frac{1}{2}\sin^{-1}(-\beta'/\gamma')$ and $\pi/2 - \frac{1}{2}\sin^{-1}(-\beta'/\gamma)$. They correspond to the mixed modes and, as β' goes for instance from $-\gamma'$ to γ' (if $\gamma' > 0$), one finds the same bifurcation structure as found for the original Eqs. (4), as explained formerly under the heading ''*Scenario II*'' (Fig. 2).

In this Rapid Communication, we have presented a simple model for systems made of two symmetric coupled self-oscillators [9]. This might be a theory for one of the most studied instabilities in real fluid mechanics, the periodic release of vortices in the wake of cylinders, a phenomenon studied experimentally and theoretically long ago by Bénard and von Karman [10,11] and their collaborators. The connection of the present work with the Bénard–von Karman phenomenon could be as follows. Our idea is that the wake is created by two symmetrically coupled self-oscillators, one on each side of the cylinder. We have shown that, depending on the coupling, these two systems may either oscillate in phase or out of phase (as in the Bénard–von Karman wake in a normal viscous fluid). Moreover, the transition from one of these two states to the other is realized by two different scenarios depending of the parameters. This might describe recent experiments by Le Gal and collaborators [6], who observe this transition when the flow around the cylinder is more and more constrained by plates perpendicular to the axis of this cylinder.

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